

CASTELNUOVO-MUMFORD REGULARITY AND GORENSTEINNESS OF FIBER CONE

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ABSTRACT. In this article, we study the Castelnuovo-Mumford regularity and Gorenstein properties of the fiber cone. We obtain upper bounds for the Castelnuovo-Mumford regularity of the fiber cone and obtain sufficient conditions for the regularity of the fiber cone to be equal to that of the Rees algebra. We obtain a formula for the canonical module of the fiber cone and use it to study the Gorenstein property of the fiber cone.

1. INTRODUCTION

Let (A, \mathfrak{m}) be a commutative Noetherian local ring and I be an ideal of A . The graded algebras, the Rees algebra, $R(I) := \bigoplus_{n \geq 0} I^n t^n \subset R[t]$, the associated graded ring, $G(I) := \bigoplus_{n \geq 0} I^n / I^{n+1} \cong R(I) / IR(I)$ and the fiber cone, $F(I) := \bigoplus_{n \geq 0} I^n / \mathfrak{m} I^n$ are together known as the blowup algebras associated to I . In this article, our aim is to study the Castelnuovo-Mumford regularity and the Gorensteinness of the fiber cone. We do this by relating them with the corresponding properties of certain other graded modules. Cortadellas and Zarzuela, in a series papers, used certain graded modules associated to filtrations of modules to study the depth properties of the fiber cone [CZ2], [CZ1], [C]. We use these graded modules and other blowup algebras to study the regularity and the Gorensteinness of the fiber cone.

For a standard graded algebra $S = \bigoplus_{n \geq 0} S_n$ over a commutative Noetherian ring S_0 and a finitely generated graded S -module $M = \bigoplus_{n \geq 0} M_n$, define

$$a(M) := \begin{cases} \max\{n \mid M_n \neq 0\} & \text{if } M \neq 0 \\ -\infty & \text{if } M = 0. \end{cases}$$

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For $i \geq 0$, set

$$a_i(M) := a(H_{S_+}^i(M)),$$

where S_+ denotes the ideal of S generated by the homogeneous elements of positive degree and $H_{S_+}^i(M)$ denotes the i -th local cohomology module of M with respect to the ideal S_+ . The *Castelnuovo-Mumford regularity* (or *regularity*) of M is defined as the number

$$\text{reg}(M) := \max\{a_i(M) + i \mid i \geq 0\}.$$

Let (A, \mathfrak{m}) be a local ring and I be any ideal. The Castelnuovo-Mumford regularity of $R(I)$ and $G(I)$ have been well studied in the past. Ooishi proved that $\text{reg } R(I) = \text{reg } G(I)$, [O] (see also [T]). In [T], Trung studied the vanishing behavior of the local cohomology modules of the associated graded ring and the Rees algebra and derived that for any ideal in a Noetherian local ring $\text{reg } R(I) = \text{reg } G(I)$. It can easily be seen that such an equality is not true in the case of the fiber cone and the associated graded ring (see Section 2). In Section 2, we prove that for any ideal of analytic spread one in a Noetherian local ring, the regularity of the fiber cone is bounded above by the regularity of the associated graded ring. If the ideal contains a regular element, we show that the equality holds in the above case (Theorem 2.2). We also prove that, under some assumptions, the regularity of the fiber cone is bounded below by the regularity of the associated graded ring and obtain certain sufficient conditions for the equality.

In Section 3, we study the Gorenstein property of the fiber cone. The Gorenstein property of the Rees algebra and the associated graded ring has been very well studied, see for example [GN], [HRS], [HKU], [Hy], [TVZ]. It is known that the Gorenstein fiber cones behave differently. For example, it is known that, unlike in the case of associated graded ring, fiber cone can be Gorenstein without the ambient ring being Gorenstein. Also, it can easily be seen that the symmetry of the Hilbert series of the fiber cone does not assure the Gorensteinness for fiber cone ([JPV, Example 6.2]). In Section 3, we obtain an expression for the canonical module of the fiber cone (Proposition 3.1). Using the structure of the canonical module, we obtain a necessary and sufficient condition for the Gorenstein property of the fiber cone (Theorem 3.5). We also obtain an upper bound for the regularity of the canonical module of $F(I)$. We end the article by proving that the multiplicity of the canonical module of the fiber cone is strictly less than the multiplicity of the canonical module of the associated graded ring, except for the maximal ideal. Throughout this article, (A, \mathfrak{m}) will always denote a Noetherian local ring of dimension d

and with infinite residue field. All the computations in this article have been performed using the computer algebra package CoCoA, [Co].

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2. CASTELNUOVO-MUMFORD REGULARITY OF THE FIBER CONE

In this section we study the regularity of the fiber cone. Unlike in the case of the associated graded ring and the Rees algebra, there is no equality between the regularities of the fiber cone and the Rees algebra. For example if A is a non-Buchsbaum ring and I is generated by a system of parameters and not a d -sequence (such a sequence exists by [Hu, Proposition 1.7]), then from Corollary 5.2 in [T] we have $\text{reg } R(I) > 0$. Since I is generated by a system of parameters $\text{reg } F(I) = 0$. Therefore $\text{reg } F(I) < \text{reg } R(I)$. We show that under some hypothesis, the regularity of the fiber cone is at most that of the Rees algebra. We also provide some sufficient conditions for the equality.

Let A be a Noetherian local ring of dimension $d > 0$ and $I \subset A$ be an ideal. Consider the filtration

$$\mathcal{F} : A \supset \mathfrak{m} \supset \mathfrak{m}I \supset \mathfrak{m}I^2 \supset \dots$$

Let $R(\mathcal{F}) := A \oplus \mathfrak{m}t \oplus \mathfrak{m}It^2 \oplus \mathfrak{m}I^2t^3 \oplus \dots$. Then $R(\mathcal{F})$ is a finitely generated graded $R(I)$ -module. Consider the exact sequences of $R(I)$ -modules:

$$(1) \quad 0 \rightarrow R(I) \rightarrow R(\mathcal{F}) \rightarrow \mathfrak{m}G(I)(-1) \rightarrow 0$$

$$(2) \quad 0 \rightarrow \mathfrak{m}G(I) \rightarrow G(I) \rightarrow F(I) \rightarrow 0$$

We use the above two exact sequences and the corresponding long exact sequence of local cohomology modules to study the vanishing properties of the local cohomology modules of the fiber cone. Throughout this section we assume that I is an ideal of analytic spread $:= \dim F(I) = \ell > 0$. We begin this section with a remark on the top local cohomology modules of $R(I)$ and $R(\mathcal{F})$.

Remark 2.1. Suppose $\underline{x} = x_1, \dots, x_\ell$ generates a minimal reduction of I . Note that $R(I)_+$ is generated radically by ℓ elements. Denote by \underline{x}^k the sequence x_1^k, \dots, x_ℓ^k . Then for all $n \in \mathbb{Z}$,

$$[H_{R(I)_+}^\ell(R(I))]_n \cong \lim_{\substack{\longrightarrow \\ k}} \frac{I^{\ell k+n}}{(\underline{x}^k)I^{(\ell-1)k+n}} \text{ and } [H_{R(I)_+}^\ell(R(\mathcal{F}))]_n \cong \lim_{\substack{\longrightarrow \\ k}} \frac{\mathfrak{m}I^{\ell k+n-1}}{(\underline{x}^k)\mathfrak{m}I^{(\ell-1)k+n-1}}.$$

This implies that $a_\ell(R(\mathcal{F})) - 1 \leq a_\ell(R(I))$.

In the following theorem, we show that the regularity of the fiber cone is bounded above by the regularity of the associated graded ring when $\ell = 1$. For convenience, we will denote $H_{R(I)_+}^i(M)$ by $H^i(M)$ for the rest of the section.

Theorem 2.2. *Let (A, \mathfrak{m}) be a Noetherian local ring and I be an ideal of A with $\ell = 1$. Then $\text{reg } F(I) \leq \text{reg } G(I)$. Furthermore, if $\text{grade } I = 1$, then $\text{reg } F(I) = \text{reg } G(I) = r(I)$, where $r(I)$ denotes the reduction number of I .*

Proof. Since by hypothesis $\ell = 1$, I is generated by a single element up to radical. Therefore $H^i(M) = (0)$ for all $i \geq 2$ and for any finitely generated graded $R(I)$ -module M . From the exact sequence (2), we have the long exact sequence:

$$\begin{aligned} 0 &\rightarrow H^0(\mathfrak{m}G(I)) \rightarrow H^0(G(I)) \rightarrow H^0(F(I)) \\ &\rightarrow H^1(\mathfrak{m}G(I)) \rightarrow H^1(G(I)) \rightarrow H^1(F(I)) \rightarrow 0. \end{aligned}$$

It follows that $a_0(\mathfrak{m}G(I)) \leq a_0(G(I))$ and $a_1(F(I)) \leq a_1(G(I))$. From the exact sequence (1), we have the long exact sequence:

$$\begin{aligned} 0 &\rightarrow H^0(R(I)) \rightarrow H^0(R(\mathcal{F})) \rightarrow H^0(\mathfrak{m}G(I))(-1) \\ &\rightarrow H^1(R(I)) \rightarrow H^1(R(\mathcal{F})) \rightarrow H^1(\mathfrak{m}G(I))(-1) \rightarrow 0. \end{aligned}$$

Therefore $a_1(\mathfrak{m}G(I)(-1)) = a_1(\mathfrak{m}G(I)) + 1 \leq a_1(R(\mathcal{F}))$. From Remark 2.1, it follows that $a_1(\mathfrak{m}G(I)) \leq a_1(R(I))$. Since G_+ is radically generated by one element, $H^1(G(I)) \neq 0$ and hence by Theorem 3.1 of [T], we get $a_1(R(I)) \leq a_1(G(I))$ so that $a_1(\mathfrak{m}G(I)) \leq a_1(G(I))$. Therefore $\text{reg } \mathfrak{m}G(I) \leq \text{reg } G(I)$. Now the regularity behaviour under the exact sequence (2) yields that

$$\text{reg } F(I) \leq \max\{\text{reg } G(I), \text{reg } \mathfrak{m}G(I) - 1\} = \text{reg } G(I).$$

Now assume that $\text{grade } I = 1$. Then by ([CZ3], page 764) we have $\text{reg } F(I) = r_J(I)$, for any minimal reduction J of I . Also we have $\text{reg } G(I) = r_J(I)$ (see for example ([HZ], Proposition 3.6)). Therefore $\text{reg } F(I) = \text{reg } G(I) = r(I)$. \square

Now we give a lower bound for the regularity of fiber cone under some assumptions. In [CZ3], Cortadellas and Zarzuela proved that if the depth of the fiber cone and the associated graded ring is at least $\ell - 1$, then the regularities of these two algebras are

equal. We generalize this result in the following theorem and retrieve their result in the above mentioned case. For $x \in I \setminus \mathfrak{m}I$, let x^* denote the image of x in I/I^2 and x^o denote the image of x in $I/\mathfrak{m}I$.

Theorem 2.3. *Let (A, \mathfrak{m}) be a Noetherian local ring and I be an ideal of A . Suppose $\text{grade } I = \ell$ and $\text{grade } G(I)_+ \geq \ell - 1$. Then $\text{reg } F(I) \geq \text{reg } G(I)$. Furthermore, if $\text{depth } F(I) \geq \ell - 1$, then $\text{reg } F(I) = \text{reg } G(I)$.*

Proof. If $\ell = 1$, then the proposition follows from Theorem 2.2. Suppose $\ell \geq 2$. Let x_1, \dots, x_ℓ be a minimal generating set for a minimal reduction J of I such that $x_1^*, \dots, x_\ell^* \in I/I^2$ is a filter regular sequence for $G(I)$ and $x_1^o, \dots, x_\ell^o \in I/\mathfrak{m}I$ is a filter regular sequence for $F(I)$. Since $\text{grade } G(I)_+ \geq \ell - 1$, $x_1^*, \dots, x_{\ell-1}^*$ is $G(I)$ -regular. Let “-” denote modulo $(x_1, \dots, x_{\ell-1})$. Then $G(\bar{I}) \cong G(I)/(x_1^*, \dots, x_{\ell-1}^*)$, $F(\bar{I}) \cong F(I)/(x_1^o, \dots, x_{\ell-1}^o)$ and $\text{reg } G(\bar{I}) = \text{reg } G(I)$. Since $\dim F(\bar{I}) = 1$, by Theorem 2.2 we get $\text{reg } F(\bar{I}) = \text{reg } G(\bar{I}) = \text{reg } G(I)$. From [CH, Proposition 1.2], it follows that $\text{reg } F(I) \geq \text{reg } F(\bar{I})$. This implies that $\text{reg } F(I) \geq \text{reg } G(I)$.

Now assume $\text{depth } F(I) \geq \ell - 1$. Then $x_1^o, \dots, x_{\ell-1}^o$ is $F(I)$ -regular. Then $\text{reg } F(I) = \text{reg } F(\bar{I})$. Therefore $\text{reg } F(I) = \text{reg } G(I)$ as required. \square

Now we give certain instances where the regularity of the fiber cone is equal to the regularity of the Rees algebra or the associated graded ring.

Proposition 2.4. *Let (A, \mathfrak{m}) be a Noetherian local ring and I be an ideal of A . If $\text{reg } R(\mathcal{F}) \leq \text{reg } R(I)$, then $\text{reg } F(I) = \text{reg } R(I)$.*

Proof. From the exact sequence (1) and the fact that $\text{reg } \mathfrak{m}G(I)(-1) = \text{reg } \mathfrak{m}G(I) + 1$, it follows that $\text{reg } \mathfrak{m}G(I) + 1 \leq \max\{\text{reg } R(I) - 1, \text{reg } R(\mathcal{F})\}$. Since $\text{reg } R(\mathcal{F}) \leq \text{reg } R(I)$, the above inequality implies that $\text{reg } \mathfrak{m}G(I) + 1 \leq \text{reg } R(I)$. From the exact sequence (2), we get $\text{reg } F(I) \leq \max\{\text{reg } \mathfrak{m}G(I) - 1, \text{reg } G(I)\}$. Since $\text{reg } R(I) = \text{reg } G(I)$, the above inequality implies that $\text{reg } F(I) \leq \text{reg } R(I)$.

Now from the exact sequence (2), $\text{reg } G(I) \leq \max\{\text{reg } \mathfrak{m}G(I), \text{reg } F(I)\}$. Since $\text{reg } G(I) = \text{reg } R(I)$ and $\text{reg } \mathfrak{m}G(I) \leq \text{reg } R(I) - 1$, the above inequality yields that

$$\max\{\text{reg } \mathfrak{m}G(I), \text{reg } F(I)\} = \text{reg } F(I) \text{ and } \text{reg } R(I) = \text{reg } G(I) \leq \text{reg } F(I).$$

Therefore we have $\text{reg } R(I) = \text{reg } F(I)$. This completes the proof. \square

Note that, if $\text{grade}_{R(I)_+} R(\mathcal{F}) \geq \ell$, then $H^i(R(\mathcal{F})) = 0$ for $i < \ell$ and from Remark 2.1, it follows that $\text{reg } R(\mathcal{F}) \leq \text{reg } R(I)$. The next proposition gives yet another instance of the equality of the regularity of these graded algebras.

Proposition 2.5. *Let (A, \mathfrak{m}) be a Noetherian local ring and I be an \mathfrak{m} -primary ideal of A such that $\text{grade } I > 0$. Suppose $I^{n_0} = \mathfrak{m}I^{n_0-1}$ for some $n_0 \in \mathbb{N}$. Then $\text{reg } F(I) = \text{reg } G(I)$.*

Proof. Since $I^{n_0} = \mathfrak{m}I^{n_0-1}$ for some $n_0 \in \mathbb{N}$, it follows that $\mathfrak{m}G(I)$ is Artinian. If $I = \mathfrak{m}$, then the assertion of the theorem follows trivially. If $I \neq \mathfrak{m}$, then $\mathfrak{m}G(I) \neq 0$. Therefore $H^0(\mathfrak{m}G(I)) = \mathfrak{m}G(I) \neq 0$ and $H^i(\mathfrak{m}G(I)) = 0$ for all $i > 0$. From the exact sequence (2) we have

$$0 \rightarrow H^0(\mathfrak{m}G(I)) \rightarrow H^0(G(I)) \rightarrow H^0(F(I)) \rightarrow 0$$

and $H^i(G(I)) \cong H^i(F(I))$ for all $i > 0$. Therefore $a_0(F(I)) \leq a_0(G(I))$ and $a_i(G(I)) = a_i(F(I))$ for $i > 0$. If $\text{depth } G(I) > 0$, then $H^0(G(I)) = (0)$ and hence from the above exact sequence we get $H^0(\mathfrak{m}G(I)) = 0$, which is a contradiction. Therefore $\text{depth } G(I) = 0$. Then by the Proposition 6.1 in [T], we have

$$a_0(F(I)) \leq a_0(G(I)) < a_1(G(I)) = a_1(F(I)).$$

Therefore $\text{reg } F(I) = \max\{a_i(F(I)) + i : i \geq 1\} = \max\{a_i(G(I)) + i : i \geq 1\} = \text{reg } G(I)$ as required. \square

Proposition 2.6. *Let (A, \mathfrak{m}) be a Noetherian local ring and I be an ideal of A such that $\text{grade } I > 0$. Assume that $\mathfrak{m}G(I)$ is a Cohen-Macaulay $R(I)$ -module of dimension ℓ . Then*

- (i) $\text{reg } F(I) \leq \text{reg } R(I)$;
- (ii) if $a_\ell(R(\mathcal{F})) - 1 < a_\ell(R(I))$, then $\text{reg } F(I) = \text{reg } R(I)$;
- (iii) if $a_\ell(R(\mathcal{F})) - 1 = a_\ell(R(I))$, then $\text{reg } \mathfrak{m}G(I) \leq \text{reg } R(I)$ and $\text{reg } F(I) \leq \text{reg } R(I)$.
Furthermore, if $\text{reg } \mathfrak{m}G(I) < \text{reg } G(I)$, then $\text{reg } F(I) = \text{reg } R(I)$.

Proof. From the short exact sequence (1), there is a long exact sequence of the local cohomology modules:

$$\cdots \rightarrow H^i(R(I)) \rightarrow H^i(R(\mathcal{F})) \rightarrow H^i(\mathfrak{m}G(I)(-1)) \rightarrow H^{i+1}(R(I)) \rightarrow \cdots.$$

Since $\mathfrak{m}G(I)$ is Cohen-Macaulay, $H^i(\mathfrak{m}G(I)) = 0$ for $i \neq \ell$. Therefore, it follows from the above long exact sequence that $H^i(R(I)) \cong H^i(R(\mathcal{F}))$ for $i < \ell$. This implies that

$a_i(R(\mathcal{F})) = a_i(R(I))$ for $i < \ell$. Using Remark 2.1, we have $a_\ell(R(\mathcal{F})) - 1 \leq a_\ell(R(I))$. Therefore (i) follows from (ii) and (iii).

If $a_\ell(R(\mathcal{F})) - 1 < a_\ell(R(I))$, then $a_\ell(R(\mathcal{F})) \leq a_\ell(R(I))$. Therefore $a_i(R(\mathcal{F})) \leq a_i(R(I))$ for all i . This implies that $\text{reg } R(\mathcal{F}) \leq \text{reg } R(I)$. Therefore from the Proposition 2.4 we have $\text{reg } F(I) = \text{reg } R(I)$. This proves (ii).

Suppose $a_\ell(R(\mathcal{F})) - 1 = a_\ell(R(I))$. Since $a_i(R(\mathcal{F})) = a_i(R(I))$ for $i < \ell$, we have $\text{reg } R(\mathcal{F}) \leq \text{reg } R(I) + 1$. Therefore from the exact sequence (1), it follows that

$$\text{reg}(\mathfrak{m}G(I)(-1)) = \text{reg } \mathfrak{m}G(I) + 1 \leq \max\{\text{reg } R(I) - 1, \text{reg } R(\mathcal{F})\} \leq \text{reg } R(I) + 1.$$

From the exact sequence (2), it follows that $\text{reg } F(I) \leq \max\{\text{reg } \mathfrak{m}G(I) - 1, \text{reg } G(I)\} \leq \text{reg } R(I)$. Thus $\text{reg } F(I) \leq \text{reg } R(I)$.

Now assume that $\text{reg } \mathfrak{m}G(I) < \text{reg } G(I)$. From the exact sequence (2) it follows that $\text{reg } G(I) \leq \max\{\text{reg } \mathfrak{m}G(I), \text{reg } F(I)\}$. Since $\text{reg } \mathfrak{m}G(I) < \text{reg } G(I)$, the above inequality gives that $\max\{\text{reg } \mathfrak{m}G(I), \text{reg } F(I)\} = \text{reg } F(I)$. Therefore $\text{reg } G(I) \leq \text{reg } F(I)$. Since $\text{reg } G(I) = \text{reg } R(I)$, we have $\text{reg } R(I) \leq \text{reg } F(I)$. The other inequality is already proved. Therefore $\text{reg } F(I) = \text{reg } R(I)$. This proves (iii). \square

We conclude this section by giving some examples to illustrate the regularity behavior of the fiber cone. The following example shows that the regularity of the fiber cone can be strictly less than the regularity of the associated graded ring even when $\text{reg } F(I) > 0$. Let $R = \bigoplus_{n \geq 0} R_n$ be a finitely generated standard graded algebra. Then $R \cong R_0[X_1, \dots, X_m]/J$ for some m and a homogeneous ideal J , where X_1, \dots, X_m are indeterminates over R_0 . Then the relation type of R , denoted by $\text{reltype}(R)$ is defined to be the maximum degree of a minimal generating set of J . It is known that $\text{reltype}(R) \leq \text{reg } R + 1$, [T]. Let k denote a field.

Example 2.7. Let $A = k[[X, Y, Z]]/(X^2, Y^2, XYZ^2)$ and $\mathfrak{m} = (x, y, z)$, where $x = \bar{X}, y = \bar{Y}, z = \bar{Z}$ and k is a field. Then A is a one dimensional Noetherian local ring. Let $I = (y, z)A$. Then $F(I) \cong k[Y, Z]/(Y^2)$. Therefore $\text{reg } F(I) = 1$. Let $\psi : A/I[U, V] \rightarrow G(I)$ be A/I -algebra homomorphism defined by $\psi(U) = yt$ and $\psi(V) = zt$. Then $\ker(\psi)$ has a generator $(x + I)UV^2$. Therefore $\text{reltype } G(I) \geq 3$. Since $\text{reltype } G(I) \leq \text{reg } G(I) + 1$ we have $\text{reg } G(I) \geq 2$. Thus $\text{reg } F(I) < \text{reg } G(I)$. Here $\text{grade } I = 0$.

Example 2.8. Let $A = k[[x, y, z]]/J$, where

$$J = (xy^3, xy^2z, xyz^2, xz^3, x^3z^2, x^4, y^3z, x^3y, x^2y^2, y^4)$$

and k is a field. Let $I = (\bar{x}^2, \bar{y}, \bar{z})$ and $\mathfrak{m} = (\bar{x}, \bar{y}, \bar{z})$. Then A is one dimensional non-Cohen-Macaulay Noetherian local ring and I is an \mathfrak{m} -primary ideal satisfying $I^4 = \mathfrak{m}I^3$. Let $S = k[U, V, W]$. Then $F(I) \cong \frac{S}{(U^2, UV^2, V^3W, UVW^2, UW^3, V^4)}$. By using CoCoA, [Co], one can see that the minimal free resolution of $F(I)$ as an S -module is

$$0 \rightarrow S(-6)^3 \rightarrow S(-4) \oplus S(-5)^7 \rightarrow S(-2) \oplus S(-3) \oplus S(-4)^4 \rightarrow S \rightarrow 0.$$

From this the regularity of $F(I)$ is 3. Therefore by the Proposition 2.5 we have $\text{reg } F(I) = \text{reg } G(I) = 3$. Note that $\text{grade } I = 0$.

The following example shows that the reduction number of I can be strictly smaller than the regularity of $F(I)$.

Example 2.9. Let $A = k[[X, Y, Z]]/J$, where $J = (X^4, XY^2Z, XYZ^2, YZ^4, Z^5)$. Let $I = (x^3, y^2, z^2)$ and $\mathfrak{m} = (x, y, z)$, where $x = \bar{X}, y = \bar{Y}$ and $z = \bar{Z}$. Then A is a non-Cohen-Macaulay Noetherian local ring. Let $J = (y^2)$, then $I^4 = JI^3$. Therefore $r(I) \leq 3$. Let $S = k[U, V, W]$. Then $F(I) \cong \frac{S}{(U^2, UVW, W^3, V^2W^2, V^4W)}$. By using CoCoA, [Co], one can see that the minimal free resolution of $F(I)$ as S -module is

$$\begin{aligned} 0 &\longrightarrow R(-6)^2 \oplus R(-7) \longrightarrow R(-4) \oplus R(-5)^4 \oplus R(-6)^2 \\ &\longrightarrow R(-2) \oplus R(-3)^2 \oplus R(-4) \oplus R(-5) \longrightarrow R \longrightarrow 0. \end{aligned}$$

Hence $\text{reg}(F(I)) = 4$. Therefore $r(I) < \text{reg } F(I)$.

Though we have proved that under certain conditions, $\text{reg } F(I)$ is bounded below by $\text{reg } G(I)$, we have not been able to find an example when this inequality is strict. Therefore we ask:

Question 2.10. Let (A, \mathfrak{m}) be a Noetherian local ring with infinite residue field and I an ideal such that $\text{grade } I > 0$. Is $\text{reg } F(I) \leq \text{reg } G(I)$?

3. GORENSTEIN FIBER CONES

In this section, we study the Gorenstein property of the fiber cone. We begin by obtaining an expression for the canonical module of the fiber cone. We fix the notation

for this section. Throughout this section, we assume that $G(I)$ and $F(I)$ are Cohen-Macaulay. Let $\omega_{G(I)}$ and $\omega_{F(I)}$ denote the canonical modules of the associated graded ring $G(I)$ and the fiber cone $F(I)$ respectively. For the definition and basic properties of canonical modules, see [BH]. In the original manuscript, the result given below was proved in a weaker form. We would like to thank the referee for suggesting the following improved form.

Proposition 3.1. *Let (A, \mathfrak{m}) be a Noetherian local ring and I be an \mathfrak{m} -primary ideal such that the associated graded ring $G(I)$ is Cohen-Macaulay. Let $\omega_{G(I)} = \bigoplus_{n \in \mathbb{Z}} \omega_n$ and $\omega_{F(I)}$ be the canonical modules of $G(I)$ and $F(I)$ respectively. Then*

- (1) $\omega_{F(I)} \cong \bigoplus_{n \in \mathbb{Z}} (0 :_{\omega_n} \mathfrak{m})$;
- (2) $a(F(I)) = a(G(I)) = r - d$, where r is the reduction number of I with respect to any minimal reduction J of I ;
- (3) for any $k \in \mathbb{N}$, $a(F(I^k)) = \lfloor \frac{a(F(I))}{k} \rfloor = \lfloor \frac{r-d}{k} \rfloor$;
- (4) if $G(I)$ is Gorenstein, then

$$\omega_{F(I)} \cong \bigoplus_{n \in \mathbb{Z}} \frac{(I^{n+r-d+1} : \mathfrak{m}) \cap I^{n+r-d}}{I^{n+r-d+1}}.$$

Proof. (1) Since $G(I)$ is Cohen-Macaulay and $F(I) = G(I)/\mathfrak{m}G(I)$ is such that $\dim G(I) = \dim F(I) = d$ we have by ([HIO], Corollary (36.14)) that:

$$\omega_{F(I)} \cong \text{Hom}_{G(I)}(F(I), \omega_{G(I)}) \cong (0 :_{\omega_{G(I)}} \mathfrak{m}G(I)) = (0 :_{\omega_{G(I)}} \mathfrak{m}) = \bigoplus_{n \in \mathbb{Z}} (0 :_{\omega_n} \mathfrak{m}).$$

(2) By definition $a(F(I)) = -\min\{n \mid [\omega_{F(I)}]_n \neq 0\}$. Since ω_n is a finitely generated A/I -module for any n and I is \mathfrak{m} -primary, A/I is of finite length and so ω_n . As a consequence $(0 :_{\omega_n} \mathfrak{m}) \neq 0$ if and only if $\omega_n \neq 0$. Therefore $a(F(I)) = a(G(I))$.

On the other hand, it is known (see for instance ([HZ], Proposition 3.6)) that if $G(I)$ is Cohen-Macaulay then $a(G(I)) = r - d$, where $r := r_J(I)$ is the reduction number of I with respect to any minimal reduction J of I .

(3) Since $G(I)$ is Cohen-Macaulay, $G(I^k)$ is also Cohen-Macaulay for any positive integer k (see for instance ([HZ], Corollary 4.6)). On the other hand, by ([HZ], Corollary 4.6) $a(G(I^k)) = \lfloor \frac{a(G(I))}{k} \rfloor$. Thus by (2)

$$a(F(I^k)) = a(G(I^k)) = \lfloor \frac{a(G(I))}{k} \rfloor = \lfloor \frac{a(F(I))}{k} \rfloor = \lfloor \frac{r-d}{k} \rfloor.$$

(4) If $G(I)$ is Gorenstein then $\omega_{G(I)} \cong G(I)(a(G(I))) = G(I)(r - d)$. So $\omega_n = I^{n+r-d}/I^{n+r-d+1}$ for any n and by (1)

$$\omega_{F(I)} \cong \bigoplus_{n \in \mathbb{Z}} (0 :_{\omega_n} \mathfrak{m}) = \bigoplus_{n \in \mathbb{Z}} (I^{n+r-d+1} : \mathfrak{m}) \cap I^{n+r-d}/I^{n+r-d+1}.$$

□

We know that if $G(I)$ and $F(I)$ are Cohen-Macaulay then $\text{reg } G(I) = r = \text{reg } F(I)$. Now we prove that $\text{reg } \omega_{G(I)}$ and $\text{reg } \omega_{F(I)}$ are equal.

Corollary 3.2. *Suppose $G(I)$ and $F(I)$ are Cohen-Macaulay. Then $\text{reg } \omega_{G(I)} = \text{reg } \omega_{F(I)}$. In addition if $G(I)$ or $F(I)$ is Gorenstein then $\text{reg } \omega_{G(I)} = d = \text{reg } \omega_{F(I)}$.*

Proof. Let J be a minimal reduction of I minimally generated by x_1, \dots, x_d such that $x_1^*, \dots, x_d^* \in G(I)$ and $x_1^o, \dots, x_d^o \in F(I)$ are regular sequences. Then $F(I)/J^o \cong F(I/J)$ and $G(I)/J^* \cong G(I/J)$. Therefore $\text{reg } \omega_{F(I)} = \text{reg}(\omega_{F(I)}/J^o \omega_{F(I)}) = \text{reg } \omega_{F(I/J)}$ and $\text{reg } \omega_{G(I)} = \text{reg } \omega_{G(I/J)}$. But $\text{reg } \omega_{F(I/J)} = a(\omega_{F(I/J)})$ and $\text{reg } \omega_{G(I/J)} = a(\omega_{G(I/J)})$. By Proposition 3.1(1) we have $\omega_{F(I/J)} \cong \bigoplus_{n \in \mathbb{Z}} (0 :_{[\omega_{G(I/J)}]_n} \mathfrak{m})$. From this we have $[\omega_{F(I/J)}]_n \neq 0$ if and only if $[\omega_{G(I/J)}]_n \neq 0$ for any n . Therefore $a(\omega_{F(I/J)}) = a(\omega_{G(I/J)})$. Thus $\text{reg } \omega_{G(I)} = \text{reg } \omega_{F(I)}$.

Now assume $G(I)$ is Gorenstein. Then $\omega_{G(I)} \cong G(I)(r - d)$. Thus $\text{reg } \omega_{G(I)} = \text{reg } G(I)(r - d) = \text{reg } G(I) - r + d$. Since $G(I)$ is Cohen-Macaulay $\text{reg } G(I) = r$. Therefore $\text{reg } \omega_{G(I)} = d$. If $F(I)$ is Gorenstein, then one can prove the statement in a similar manner. □

Corollary 3.3. *Let (A, \mathfrak{m}) be a Gorenstein local ring of dimension $d > 0$ and I is an \mathfrak{m} -primary ideal with $G(I)$ is Cohen-Macaulay. Then $\omega_{F(I)} \cong \bigoplus_{n \in \mathbb{Z}} \frac{(J^{n+r-d+1} : \mathfrak{m}I^r) \cap (J^{n+r-d} : I^r)}{(J^{n+r-d+1} : I^r)}$. If in addition $F(I)$ is Gorenstein, then*

$$\lambda \left(\frac{(J^{n+1} : \mathfrak{m}I^r) \cap (J^n : I^r)}{(J^{n+1} : I^r)} \right) = \lambda \left(\frac{I^n}{\mathfrak{m}I^n} \right)$$

for all $n \in \mathbb{Z}^+$.

Proof. By ([HKU], Theorem 4.1), $\omega_B = \bigoplus_{n \in \mathbb{Z}} (J^{n+r} : I^r) t^{n+d-1}$, where $B = A[It, t^{-1}]$ the extended Rees algebra of I . Since $\omega_{G(I)} = \omega_{B/t^{-1}B} = (\omega_B/t^{-1}\omega_B)(-1)$, we have

$\omega_{G(I)} = \bigoplus_{n \in \mathbb{Z}} \frac{(J^{n+r-d}; I^r)}{(J^{n+r-d+1}; I^r)}$. From Proposition 3.1(1),

$$\begin{aligned} \omega_{F(I)} &= \bigoplus_{n \in \mathbb{Z}} (0 :_{[\omega_{G(I)}]_n} \mathfrak{m}) \\ &= \bigoplus_{n \in \mathbb{Z}} \frac{(J^{n+r-d+1} : \mathfrak{m}I^r) \cap (J^{n+r-d} : I^r)}{(J^{n+r-d+1} : I^r)} \end{aligned}$$

Assume $F(I)$ is Gorenstein. Then $\omega_{F(I)} \cong F(I)(r-d) = \bigoplus_{n \in \mathbb{Z}} I^{n+r-d}/\mathfrak{m}I^{n+r-d}$. Therefore

$$\bigoplus_{n \in \mathbb{Z}} \frac{(J^{n+r-d+1} : \mathfrak{m}I^r) \cap (J^{n+r-d} : I^r)}{(J^{n+r-d+1} : I^r)} = \omega_{F(I)} = \bigoplus_{n \in \mathbb{Z}} I^{n+r-d}/\mathfrak{m}I^{n+r-d}.$$

This implies that

$$\lambda \left(\frac{(J^{n+1} : \mathfrak{m}I^r) \cap (J^n : I^r)}{(J^{n+1} : I^r)} \right) = \lambda \left(\frac{I^n}{\mathfrak{m}I^n} \right)$$

for all $n \in \mathbb{Z}^+$. □

Remark 3.4. Suppose $G(I)$ and $F(I)$ are Cohen-Macaulay rings. Let $J = (x_1, \dots, x_d)$ be a minimal reduction of I such that $x_1^*, \dots, x_d^* \in G(I)$ and $x_1^o, \dots, x_d^o \in F(I)$ are superficial sequences and hence regular sequences. Denote $J_i = (x_1, \dots, x_i)$ and $J_0 = (0)$. Then for any i such that $1 \leq i \leq d$, $G(I)/(x_1^*, \dots, x_i^*) \cong G(I/J_i)$ and $F(I)/(x_1^o, \dots, x_i^o) \cong F(I/J_i)$. Then $G(I)$ is Gorenstein if and only if $G(I/J_i)$ is Gorenstein and $F(I)$ is Gorenstein if and only if $F(I/J_i)$ is Gorenstein.

Now we give a characterization for $F(I)$ to be Gorenstein in terms of certain length conditions involving a minimal reduction of I if $G(I)$ is Gorenstein.

Theorem 3.5. Let (A, \mathfrak{m}) be a Noetherian local ring, I be an \mathfrak{m} -primary ideal and J be a minimal reduction of I with reduction number r . Assume that $G(I)$ is a Gorenstein ring and $F(I)$ is Cohen-Macaulay. Then $F(I)$ is Gorenstein if and only if

$$\lambda \left(\frac{((I^{n+1} + J) : \mathfrak{m}) \cap I^n}{I^{n+1} + JI^{n-1}} \right) = \lambda \left(\frac{I^n}{\mathfrak{m}I^n + JI^{n-1}} \right)$$

for $0 \leq n \leq r$.

Proof. Since $G(I)$ and $F(I)$ are Cohen-Macaulay, we may choose a generating set for J such that the corresponding images form a regular sequence in $G(I)$ as well as in $F(I)$.

Therefore we have

$$\begin{aligned}\omega_{F(I/J)} &= (\omega_{F(I)}/J^o\omega_{F(I)})(r) \\ &= \bigoplus_{n \in \mathbb{Z}} \left[\frac{((I^{n+r+1} + J) : \mathfrak{m}) \cap I^{n+r} + J}{I^{n+r+1} + J} \right].\end{aligned}$$

Using the isomorphism theorems and the fact that $G(I)$ is Cohen-Macaulay, one obtains the isomorphism:

$$[\omega_{F(I/J)}]_{n-r} \cong \frac{((I^{n+1} + J) : \mathfrak{m}) \cap I^n}{I^{n+1} + JI^{n-1}}.$$

Suppose $F(I)$ is Gorenstein. Then from the Remark 3.4, it follows that $F(I/J)$ is Gorenstein with canonical module $\omega_{F(I/J)} = F(I/J)(r)$. Therefore

$$\begin{aligned}\lambda \left(\frac{((I^{n+1} + J) : \mathfrak{m}) \cap I^n}{I^{n+1} + JI^{n-1}} \right) &= \lambda([\omega_{F(I/J)}]_{n-r}) \\ &= \lambda([F(I/J)(r)]_{n-r}) = \lambda \left(\frac{I^n}{\mathfrak{m}I^n + JI^{n-1}} \right)\end{aligned}$$

for all n . Hence, in particular, we get the required equality of lengths for $0 \leq n \leq r$.

Conversely assume that $\lambda(((I^{n+1} + J) : \mathfrak{m}) \cap I^n / I^{n+1} + JI^{n-1}) = \lambda(I^n / \mathfrak{m}I^n + JI^{n-1})$ for $0 \leq n \leq r$. Then by the above isomorphisms we have $\lambda([\omega_{F(I/J)}]_n) = \lambda([F(I/J)(r)]_n)$ for all n . This implies that $\lambda(\omega_{F(I/J)}) = \lambda(F(I/J)(r)) = \lambda(F(I/J))$. Let $\eta : P \rightarrow \omega_{F(I/J)}$ be the natural surjective map from a graded free $F(I/J)$ -module P of rank equal to $\mu(\omega_{F(I/J)})$, the minimal number of generators of $\omega_{F(I/J)}$. Since $\omega_{F(I/J)}$ has finite injective dimension, its injective dimension is equal to $\text{depth } F(I/J) = 0$. Therefore $\omega_{F(I/J)}$ is an injective module and hence $\text{Hom}_{F(I/J)}(-, \omega_{F(I/J)})$ is an exact functor. Applying this exact functor to the map η , we get a surjective map $\eta^* : \text{Hom}_{F(I/J)}(\omega_{F(I/J)}, \omega_{F(I/J)}) \rightarrow \text{Hom}_{F(I/J)}(P, \omega_{F(I/J)})$. But by the definition of canonical module,

$$\text{Hom}_{F(I/J)}(\omega_{F(I/J)}, \omega_{F(I/J)}) \cong F(I/J).$$

Note that

$$\text{Hom}_{F(I/J)}(P, \omega_{F(I/J)}) \cong \bigoplus_{\text{rank}(P)} \omega_{F(I/J)}.$$

Hence there is a surjective map $F(I/J) \rightarrow \bigoplus_{\text{rank}(P)} \omega_{F(I/J)}$. This implies that

$$\lambda \left(\bigoplus_{\text{rank}(P)} \omega_{F(I/J)} \right) \leq \lambda(F(I/J)).$$

This gives that $\text{rank}(P) \cdot \lambda(\omega_{F(I/J)}) \leq \lambda(F(I/J))$. Since $\lambda(\omega_{F(I/J)}) = \lambda(F(I/J))$, the above inequality gives that $\text{rank}(P) = 1$. That is $\mu(\omega_{F(I/J)}) = 1$. Hence $F(I/J)$ is Gorenstein with canonical module $\omega_{F(I/J)} = F(I/J)(r)$. Since J^o is generated by a regular sequence in $F(I)$, $F(I)$ is Gorenstein with canonical module $\omega_{F(I)} = F(I)(r - d)$. \square

It is known that if $G(I)$ is Gorenstein, then A is Gorenstein and that such an analogue is not true in the case of fiber cone. We show that if, in addition, $F(I)$ is Gorenstein, then A/I is Gorenstein.

Corollary 3.6. *Let (A, \mathfrak{m}) be a Noetherian local ring and I is an \mathfrak{m} -primary ideal. Suppose $G(I)$ and $F(I)$ are Gorenstein. Then A/I is Gorenstein.*

Proof. Put $n = 0$ in the Theorem 3.5, we get $\lambda((I : \mathfrak{m})/I) = \lambda(A/\mathfrak{m}) = 1$. This implies that A/I is Gorenstein. \square

Let $\text{pd}_A(M)$ denote the projective dimension of M as an A -module.

Corollary 3.7. *Suppose I is an \mathfrak{m} -primary ideal such that $G(I)$ and $F(I)$ are Gorenstein. Then $\text{pd}(A/I) < \infty$ if and only if I is generated by a regular sequence.*

Proof. Since $G(I)$ and $F(I)$ are Gorenstein, from the Corollary 3.6 it follows that A/I is Gorenstein. Suppose $\text{pd}(A/I) < \infty$. Then from Theorem 2.6 of [NV], I is generated by a regular sequence. Conversely assume I is generated by a regular sequence. Then the Koszul complex $K_\bullet(I; A)$ is minimal free resolution of A/I . Therefore $\text{pd}(A/I) < \infty$. \square

Corollary 3.8. *Suppose (A, \mathfrak{m}) is a regular local ring and I is an \mathfrak{m} -primary ideal of A such that $G(I)$ and $F(I)$ are Gorenstein. Then I is generated by a regular sequence.*

Proof. By hypothesis A is regular, so we have $\text{pd}(A/I) < \infty$. Therefore from Corollary 3.7, I is generated by a regular sequence. \square

Remark 3.9. *Suppose $G(I)$ and $F(I)$ are Gorenstein rings and $\mu(I) = d + 2$. Then by Corollary 3.6, A/I is Gorenstein. By ([HKU], Remark 2.9(1), (2)) we have $\text{pd}_A(G(I)) < \infty$ if and only if $G(I)$ is a complete intersection. That is the defining ideal of $G(I)$ is generated by a regular sequence. In this case $F(I)$ is also a complete intersection.*

The expression we have obtained for the canonical module of the fiber cone does not reveal when it can be realized as a submodule of $F(I)$. Below we give a sufficient condition for $\omega_{F(I)}$ to be a submodule of $F(I)$.

Corollary 3.10. *Let (A, \mathfrak{m}) be a Noetherian local ring and I be an \mathfrak{m} -primary ideal. Assume that $G(I)$ is a Gorenstein ring. Suppose $F(I)$ is Cohen-Macaulay and A/I is Gorenstein. If $(I^{n+r-d+1} : \mathfrak{m}) \cap \mathfrak{m}I^{n+r-d} = I^{n+r-d}$ for all $n \geq d - r + 1$, then $\omega_{F(I)}$ is an ideal of $F(I)$.*

Proof. For any $n \geq d - r$, there is a natural map $\psi_n : (I^{n+r-d+1} : \mathfrak{m}) \cap I^{n+r-d} / I^{n+r-d} \rightarrow I^{n+r-d} / \mathfrak{m}I^{n+r-d}$ which gives rise to a natural $F(I)$ -linear map $\psi : \omega_{F(I)} \rightarrow F(I)$. The kernel of ψ_n is $(I^{n+r-d+1} : \mathfrak{m}) \cap \mathfrak{m}I^{n+r-d} / I^{n+r-d}$. Then the $F(I)$ -linear map $\psi : \omega_{F(I)} \rightarrow F(I)$ has kernel $\bigoplus_{n \geq d-r} (I^{n+r-d+1} : \mathfrak{m}) \cap \mathfrak{m}I^{n+r-d} / I^{n+r-d}$. Then $\omega_{F(I)}$ is an ideal of $F(I)$ if $\ker(\psi) = 0$, i.e., if $(I^{n+r-d+1} : \mathfrak{m}) \cap \mathfrak{m}I^{n+r-d} = I^{n+r-d}$ for all $n \geq d - r + 1$. This completes the proof. \square

The following Proposition shows that $e_0(\omega_{F(I)}) < e_0(\omega_{G(I)})$ unless $I = \mathfrak{m}$.

Proposition 3.11. *Let (A, \mathfrak{m}) be a Noetherian local ring and I be an \mathfrak{m} -primary ideal. Suppose $G(I)$ and $F(I)$ are Cohen-Macaulay. Then $e_0(\omega_{F(I)}) \leq e_0(\omega_{G(I)})$. Furthermore, if $G(I)$ is Gorenstein, then equality holds if and only if $I = \mathfrak{m}$.*

Proof. From Proposition 3.1(1), it is clear that $\lambda([\omega_{F(I)}]_n) \leq \lambda([\omega_{G(I)}]_n)$ for all n . Hence $e_0(\omega_{F(I)}) \leq e_0(\omega_{G(I)})$. Let $J = (x_1, \dots, x_d)$ be a minimal reduction of I with reduction number r such that $x_1^o, \dots, x_d^o \in F(I)$ is a regular sequence for $F(I)$, $\omega_{F(I)}$ and $x_1^*, \dots, x_d^* \in G(I)$ is $G(I)$ and $\omega_{G(I)}$ regular sequence. Then $e_0(\omega_{G(I)}) = e_0(\omega_{G(I/J)}) = \lambda(\omega_{G(I/J)})$ and $e_0(\omega_{F(I)}) = e_0(\omega_{F(I/J)}) = \lambda(\omega_{F(I/J)})$.

Now assume $e_0(\omega_{F(I)}) = e_0(\omega_{G(I)})$. Then $\lambda(\omega_{F(I/J)}) = \lambda(\omega_{G(I/J)})$. By Proposition 3.1(1) we have $\omega_{F(I/J)} \subseteq \omega_{G(I/J)}$. Therefore $\omega_{F(I/J)} = \omega_{G(I/J)}$. This implies that $(I : \mathfrak{m})/I = A/I$. This gives $\lambda((I : \mathfrak{m})/I) = \lambda(A/I)$. This implies $(I : \mathfrak{m}) = A$. That is $I = \mathfrak{m}$. The converse always holds. This completes the proof. \square

We conclude this article by providing two examples. First we give an example of an ideal whose associated graded ring is Gorenstein but the fiber cone is not.

Example 3.12. *Let $A = k[[t^4, t^9, t^{10}]]$ and $I = (t^8, t^{18}, t^{10})$, $\mathfrak{m} = (t^4, t^9, t^{10})$. Then A is a one dimensional Gorenstein local domain and I is an \mathfrak{m} -primary ideal. $J = (t^8)$ is a minimal reduction of I of reduction number 1. Then it follows from the Corollary 4.5 (5) of [HKU] that $G(I)$ is Gorenstein. Since I has reduction number 1, $F(I)$ is Cohen-Macaulay. Since $\mu(I) = 3 > \dim A + 1$, by Proposition 4.1 of [JPV], $F(I)$ is not*

Gorenstein. Also, note that

$$5 = \lambda \left(\frac{((I^2 + J) : \mathfrak{m}) \cap I}{I^2 + J} \right) \neq \lambda \left(\frac{I}{\mathfrak{m}I + J} \right) = 1.$$

In the example below we apply our result to obtain an example of a Gorenstein fiber cone.

Example 3.13 ([HKU], Example 2.5). Let $A = k[[t^4, t^9, t^{10}]]$ and $I := (t^8, t^9, t^{10})$. Then A is a one dimensional Gorenstein local domain, I is an \mathfrak{m} -primary ideal and $J = (t^8)$ is a minimal reduction of I with reduction number 2. Let $R = A/I[X, Y, Z]$. Then $G(I) \cong \frac{R}{(XZ - Y^2, wX^2 - Z^2)}$, where w is the image of t^4 in A/I . Since $XZ - Y^2, wX^2 - Z^2$ is an R -regular sequence and A/I is Gorenstein therefore $G(I)$ is Gorenstein. Since $\mathfrak{m}^n \cap J = \mathfrak{m}J I^{n-1}$ for all $n \geq 1$, $F(I)$ is Cohen-Macaulay. By using CoCoA, [Co], it can easily be seen that

$$\lambda \left(\frac{((I^2 + J) : \mathfrak{m}) \cap I}{I^2 + J} \right) = 2 = \lambda \left(\frac{I}{\mathfrak{m}I + J} \right)$$

and

$$\lambda \left(\frac{((I^3 + J) : \mathfrak{m}) \cap I^2}{I^2 + JI} \right) = 1 = \lambda \left(\frac{I^2}{\mathfrak{m}I^2 + JI} \right).$$

Therefore by the Theorem 3.5, $F(I)$ is Gorenstein.

REFERENCES

- [BH] W. Bruns, J. Herzog, *Cohen-Macaulay rings*, Revised Edition, Cambridge Studies in Advanced Mathematics, 39. Cambridge University Press, Cambridge, 1998.
- [CH] A. Conca, J. Herzog, *Castelnuovo-Mumford regularity of products of ideals*, Collect. Math. **54** (2003), 137-152.
- [C] T. Cortadellas, *Fiber cones with almost maximal depth*, Comm. Algebra **33** (2005), no. 3, 953–963.
- [CZ1] T. Cortadellas, S. Zarzuela, *On the depth of the fiber cone of filtrations*, J. Algebra **198** (1997), no. 2, 428–445.
- [CZ2] T. Cortadellas, S. Zarzuela, *Depth formulas for certain graded modules associated to a filtration: a survey*, Geometric and combinatorial aspects of commutative algebra (Messina, 1999), 145–157, Lecture Notes in Pure and Appl. Math., 217, Dekker, New York, 2001.
- [CZ3] T. Cortadellas, S. Zarzuela, *On the structure of the fiber cone of ideals with analytic spread one*, J. Algebra **317** (2007), no. 2, 759–785.
- [Co] CoCoATeam, *CoCoA: a system for doing Computations in Commutative Algebra*. Available at <http://cocoa.dima.unige.it>.

- [GI] S. Goto and S.-I. Iai, *Embeddings of certain graded rings into their canonical modules*, J. Algebra **228** (2000), 377–396.
- [GN] S. Goto, K. Nishida, *The Cohen-Macaulay and Gorenstein Rees algebras associated to filtrations*, Mem. Amer. Math. Soc. **110** (1994), no. 526. American Mathematical Society, Providence, RI, 1994.
- [HIO] M. Herrmann, S. Ikeda, and U. Orbanz, *Equimultiplicity and blowing up*, Springer-Verlag, Berlin, 1988.
- [HZ] L. T. Hoa and S. Zarzuela, *Reduction number and a -invariant of good filtrations*, Comm. Algebra **14** (1994), no. 22, 5635–5656.
- [Hu] C. Huneke, *The theory of d -sequences and powers of ideals*, Advances in Math. **46** (1982), 249–279.
- [Hy] E. Hyry, *On the Gorenstein property of the associated graded ring of a power of an ideal*, Manuscripta Math. **80** (1993), 13–20.
- [HKU] W. Heinzer, M.-K. Kim, B. Ulrich, *The Gorenstein and complete intersection properties of associated graded rings*, J. Pure and Applied Algebra **201** (2005), 264–283.
- [HRS] M. Herrmann, J. Ribbe, P. Schenzel, *On the Gorenstein property of the form rings*, Math. Z. **213** (1993), 301–309.
- [JPV] A. V. Jayanthan, Tony J. Puthenpurakal, J. K. Verma, *On fiber cones of \mathfrak{m} -primary ideals*, Canad. J. Math. **59** (2007), no.1, 109–126.
- [NV] S. Noh and W. Vasconcelos, *The S_2 -closure of a Rees algebra*, Results. Math. **23** (1993), 149 – 162.
- [O] A. Ooishi, *Genera and arithmetic genera of commutative rings*, Hiroshima Math. J. **17** (1987), 47–66.
- [T] N. V. Trung, *The Castelnuovo regularity of the Rees algebra and the associated graded ring*, Trans. of Amer. Math. Soc. **350** (1998), no. 7, 2813–2832.
- [TVZ] N. V. Trung, D. Q. Viêt, S. Zarzuela, *When is the Rees algebra Gorenstein?*, J. Algebra **175** (1995), 137–156.

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